Journal of Applied Mechanics and Technical Physics, Vol. 48, No. 4, pp. 614–620, 2007

STRESS–STRAIN STATE OF AN ANISOTROPIC PLATE WITH AN ELLIPTIC HOLE AND THIN RIGID INCLUSIONS

V. N. Maksimenko and S. A. Zorin

UDC 539.375:629.7.02

The stress–strain state of an anisotropic plate containing an elliptic hole and thin, absolutely rigid, curvilinear inclusions is studied. General integral representations of the solution of the problem are constructed that satisfy automatically the boundary conditions on the elliptic-hole contour and at infinity. The unknown density functions appearing in the potential representations of the solution are determined from the boundary conditions at the rigid inclusion contours. The problem is reduced to a system of singular integral equations which is solved by a numerical method. The effects of the material anisotropy, the degree of ellipticity of the elliptic hole, and the geometry of the rigid inclusions on the stress concentration in the plate are studied. The numerical results obtained are compared with existing analytical solutions.

Key words: *anisotropic plate, thin rigid inclusions, stress concentrations, stress intensity factor, integral equation.*

Let an infinite rectilinearly anisotropic plate of thickness h be weakened by an elliptic hole with the contour $L_0 = \{(x/a)^2 + (y/b)^2 = 1\}$ and a set of thin, absolutely rigid inclusions shaped like smooth open curves L_j ($j = \overline{1,k}$). The curves do not intersect each other and the contour L_0. For each contour L_j, we determine the normal vector $\boldsymbol{n}(t)$ ($t \in L_j$) directed to the right when passing from the points a_j to the points b_j (Fig. 1). The plate is loaded by external forces $X_n + iY_n$ at the hole contour and the forces σ_x^∞, σ_y^∞, and τ_{xy}^∞ at infinity. Each inclusion can perform rigid-body translations and rotations:

$$u^\pm(t) + iv^\pm(t) = g_1(t) + ig_2(t) = G(t), \qquad t \in L = \bigcup_{j=1}^{k} L_j,$$
$$G(t) = c_j + i\varepsilon_j t, \qquad t \in L_j. \tag{1}$$

Here c_j is a complex constant and ε_j is the unknown or specified rotation of the rigid inclusion L_j. The plus and minus signs refer to the left and right faces of the inclusion, respectively. It is assumed that a generalized plane stress state occurs in the plate and the rotation vanishes at infinity.

The stresses in the plate can be expressed in terms of two analytical functions $\Phi_\nu(z_\nu)$ ($\nu = 1, 2$):

$$(\sigma_x, \tau_{xy}, \sigma_y) = 2\,\mathrm{Re}\Big(\sum_{\nu=1}^{2} (\mu_\nu^2, -\mu_\nu, 1)\Phi_\nu(z_\nu)\Big) \tag{2}$$

($z_\nu = x + \mu_\nu y$ and μ_ν are the roots of the corresponding characteristic equation with positive imaginary parts [1]).

We write the functions $\Phi_\nu(z_\nu)$ ($\nu = 1, 2$) as

$$\Phi_\nu(z_\nu) = \sum_{i=1}^{2} \Phi_{\nu i}(z_\nu). \tag{3}$$

Novosibirsk State Technical University, Novosibirsk 630092; kimt@ngs.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 48, No. 4, pp. 173–180, July–August, 2007. Original article submitted September 2, 2005; revision submitted August 17, 2006.

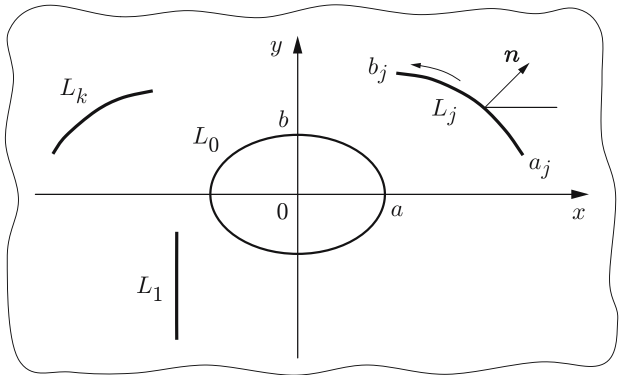

Fig. 1. Infinite anisotropic plate with an elliptic hole L_0 and a set of thin, absolutely rigid inclusions L_j $(j = \overline{1,k}\,)$.

Here $\Phi_{\nu 1}(z_\nu)$ is the solution for an infinite anisotropic plate with an elliptic hole loaded by specified forces at the hole contour and at infinity [1] and the functions $\Phi_{\nu 2}(z_\nu)$ determine the perturbed stress state due to the presence of rigid inclusions.

Using the solution of the problem of an infinite anisotropic plate which is loaded by a point force at the internal point τ and is free of stresses at the elliptic-hole contour L_0 and at infinity [1]:

$$\Psi_\nu(z_\nu, \tau_\nu) = \frac{1}{\omega_\nu'(\zeta_\nu)}\Big(\frac{A_\nu^*}{\zeta_\nu - \eta_\nu} - \frac{l_\nu \overline{A_1^*}}{\zeta_\nu(1 - \zeta_\nu \bar\eta_1)} - \frac{n_\nu \overline{A_2^*}}{\zeta_\nu(1 - \zeta_\nu \bar\eta_2)}\Big),$$

$$z_\nu = \omega_\nu(\zeta_\nu) = \frac{a - i\mu_\nu b}{2}\,\zeta_\nu + \frac{a + i\mu_\nu b}{2}\,\zeta_\nu^{-1}, \qquad |\zeta_\nu| > 1,$$

$$\zeta_\nu = \zeta_\nu(z_\nu) = \frac{z_\nu + \sqrt{z_\nu^2 - (a^2 + \mu_\nu^2 b^2)}}{a - i\mu_\nu b}, \qquad \zeta_\nu(\infty) = \infty,$$

$$\eta_\nu = \zeta_\nu(\tau_\nu) = \frac{\tau_\nu + \sqrt{\tau_\nu^2 - (a^2 + \mu_\nu^2 b^2)}}{a - i\mu_\nu b}, \qquad \tau_\nu = \operatorname{Re}\tau + \mu_\nu \operatorname{Im}\tau,$$

$$l_\nu = \frac{\mu_{3-\nu} - \bar\mu_1}{\mu_\nu - \mu_{3-\nu}}, \qquad n_\nu = \frac{\mu_{3-\nu} - \bar\mu_2}{\mu_\nu - \mu_{3-\nu}} \qquad (\nu = 1, 2)$$

(the coefficients A_ν^* depend on the magnitude and direction of the point force [1]) and using the superposition principle [2], we seek the functions $\Phi_{\nu 2}(z_\nu)$ in the form

$$\Phi_{\nu 2}(z_\nu) = \frac{1}{\omega_\nu'(\zeta_\nu)} \int\limits_L \Big(\frac{A_\nu^*(\tau)}{\zeta_\nu - \eta_\nu} - \frac{l_\nu \overline{A_1^*(\tau)}}{\zeta_\nu(1 - \zeta_\nu \bar\eta_1)} - \frac{n_\nu \overline{A_2^*(\tau)}}{\zeta_\nu(1 - \zeta_\nu \bar\eta_2)}\Big)\, ds$$

$$= \frac{1}{2\pi i \omega_\nu'(\zeta_\nu)} \int\limits_L \Big(\frac{\Omega_\nu(\tau)}{\eta_\nu - \zeta_\nu}\, d\tau_\nu - \frac{l_\nu \overline{\Omega_1(\tau)}}{\zeta_\nu(1 - \zeta_\nu \bar\eta_1)}\, d\bar\tau_1 - \frac{n_\nu \overline{\Omega_2(\tau)}}{\zeta_\nu(1 - \zeta_\nu \bar\eta_2)}\, d\bar\tau_2\Big), \tag{4}$$

$$\Omega_\nu(\tau) = -2\pi i A_\nu^*(\tau)/M_\nu(\tau), \qquad M_\nu(\tau) = \mu_\nu \cos\psi(\tau) - \sin\psi(\tau).$$

Here $\Omega_\nu(\tau) = \{\Omega_{\nu j}(\tau)\colon \tau \in L_j,\ j = \overline{1,k}\,\}$ are unknown complex functions on the contours L_j, $\psi = \psi(\tau) = \{\psi_j(\tau)\colon \tau \in L_j,\ j = \overline{1,k}\}$ is the angle between the normal vector $\boldsymbol{n}$ to the left face of the rigid inclusion L_j and the x axis, and $d\tau_\nu = M_\nu(\tau)\, ds$ (ds is an element of the arc L).

The functions $\Phi_{\nu 2}(z_\nu)$ constructed in this manner automatically satisfy the conditions $X_n = Y_n = 0$ at the hole contour L_0 and vanish at infinity. Consequently, the functions $\Phi_\nu(z_\nu)$ written in the form of (3) ensure satisfaction of the boundary conditions at the hole contour L_0 and infinity.

Differentiating expression (1) with respect to the arc length, we obtain the boundary condition [3]

$$A(t)\Phi_1^{\pm}(t_1) + B(t)\overline{\Phi_1^{\pm}(t_1)} + \Phi_2^{\pm}(t_2) = W^{\pm}(t), \qquad t \in L, \tag{5}$$

where

$$W^{\pm}(t) = \frac{\bar{p}_2\, dg_2(t)/ds - \bar{q}_2\, dg_1(t)/ds}{(\bar{p}_2 q_2 - p_2\bar{q}_2)M_2(t)}, \qquad A(t) = A_0\,\frac{M_1(t)}{M_2(t)}, \qquad B(t) = B_0\,\frac{\overline{M_1(t)}}{M_2(t)},$$

$$A_0 = \frac{\bar{p}_2 q_1 - p_1\bar{q}_2}{\bar{p}_2 q_2 - p_2\bar{q}_2}, \qquad B_0 = \frac{\bar{p}_2\bar{q}_1 - \bar{p}_1\bar{q}_2}{\bar{p}_2 q_2 - p_2\bar{q}_2},$$

$$p_\nu = a_{11}\mu_\nu^2 - a_{16}\mu_\nu + a_{12}, \qquad q_\nu = a_{12}\mu_\nu + a_{22}\mu_\nu^{-1} - a_{26},$$

$\Phi_\nu^{\pm}(t_\nu)$ are the boundary values of the functions $\Phi_\nu(z_\nu)$ on the contour L and a_{ij} are the strain coefficients of the anisotropic plate material [1].

Substituting the limit values of the functions $\Phi_\nu(z_\nu)$ from (3) and (4) into the boundary conditions at the rigid inclusions L_j (5) and performing some manipulations, we obtain

$$\int\limits_L \frac{\Omega_1(\tau)}{\eta_1 - \zeta_1}\, d\tau_1 + \int\limits_L [K_{11}(t,\tau)\Omega_1(\tau) + K_{12}(t,\tau)\overline{\Omega_1(\tau)}\,]\, ds = f^*(t), \qquad t \in L = \bigcup_{j=1}^{k} L_j. \tag{6}$$

Here

$$K_{11}(t,\tau)\, ds = \frac{\omega_1'(\zeta_1)}{2\overline{B(t)}}\Big[\frac{\overline{B(\tau)} - \overline{B(t)}}{\overline{\omega_2'(\zeta_2)}\,(\bar{\eta}_2 - \bar{\zeta}_2)}\, d\bar{\tau}_2 + \overline{B(t)}\Big(\frac{\overline{\omega_2'(\eta_2)} - \overline{\omega_2'(\zeta_2)}}{\overline{\omega_2'(\zeta_2)}\,(\bar{\eta}_2 - \bar{\zeta}_2)}\, d\bar{\eta}_2$$

$$- \frac{\omega_1'(\eta_1) - \omega_1'(\zeta_1)}{\omega_1'(\zeta_1)(\eta_1 - \zeta_1)}\, d\eta_1 + d\ln\frac{\bar{\eta}_2 - \bar{\zeta}_2}{\eta_1 - \zeta_1}\Big) - \frac{\overline{A(t)}}{\overline{\omega_1'(\zeta_1)}\,\bar{\zeta}_1}\Big(\frac{\bar{n}_1 A(\tau)}{1 - \bar{\zeta}_1\eta_2}\, d\tau_2 - \frac{\bar{l}_1}{1 - \bar{\zeta}_1\,\eta_1}\, d\tau_1\Big)$$

$$+ \frac{n_1\overline{B(\tau)}\,\overline{B(t)}}{\omega_1'(\zeta_1)\zeta_1(1 - \zeta_1\bar{\eta}_2)}\, d\bar{\tau}_2 - \frac{1}{\overline{\omega_2'(\zeta_2)}\,\bar{\zeta}_2}\Big(\frac{\bar{n}_2 A(\tau)}{1 - \bar{\zeta}_2\eta_2}\, d\tau_2 - \frac{\bar{l}_2}{1 - \bar{\zeta}_2\eta_1}\, d\tau_1\Big)\Big],$$

$$K_{12}(t,\tau)\, ds = \frac{\omega_1'(\zeta_1)}{2\overline{B(t)}}\Big[\frac{\overline{A(\tau)} - \overline{A(t)}}{\overline{\omega_2'(\zeta_2)}(\bar{\eta}_2 - \bar{\zeta}_2)}\, d\bar{\tau}_2 - \overline{A(t)}\Big(\frac{\overline{\omega_1'(\eta_1)} - \overline{\omega_1'(\zeta_1)}}{\overline{\omega_1'(\zeta_1)}(\bar{\eta}_1 - \bar{\zeta}_1)}\, d\bar{\eta}_1$$

$$- \frac{\overline{\omega_2'(\eta_2)} - \overline{\omega_2'(\zeta_2)}}{\overline{\omega_2'(\zeta_2)}(\bar{\eta}_2 - \bar{\zeta}_2)}\, d\bar{\eta}_2 + d\ln\frac{\bar{\eta}_1 - \bar{\zeta}_1}{\bar{\eta}_2 - \bar{\zeta}_2}\Big) - \frac{\overline{B(t)}}{\omega_1'(\zeta_1)\zeta_1}\Big(\frac{l_1}{1 - \zeta_1\bar{\eta}_1}\, d\tau_1 - \frac{n_1\overline{A(\tau)}}{1 - \zeta_1\bar{\eta}_2}\, d\bar{\tau}_2\Big)$$

$$- B(\tau)\, d\tau_2\Big(\frac{\bar{n}_1\overline{A(t)}}{\overline{\omega_1'(\zeta_1)}\bar{\zeta}_1(1 - \bar{\zeta}_1\eta_2)} + \frac{\bar{n}_2}{\overline{\omega_2'(\zeta_2)}\bar{\zeta}_2(1 - \bar{\zeta}_2\eta_2)}\Big)\Big],$$

$$f^*(t) = \frac{\pi i\omega_1'(\zeta_1)}{2\overline{B(t)}}\,\{\overline{W_2(t)} - 2[\overline{A(t)}\,\overline{\Phi_{11}(t_1)} + \overline{B(t)}\,\Phi_{11}(t_1) + \overline{\Phi_{21}(t_2)}\,]\},$$

$$W_2(t) = W^+(t) + W^-(t).$$

According to the assumptions of the smoothness of L_j $(j = \overline{1,k}\,)$, the functions $K_{11}(t,\tau)$, $K_{12}(t,\tau)$, and $f^*(t)$ are continuous.

The singular integral equation (6) should be supplemented by the equations

$$\int\limits_{L_j} \Omega_1(\tau)\, d\tau_1 = 0 \qquad (j = \overline{1,k}), \tag{7}$$

which require that the principal vector of all forces acting on each rigid inclusion vanish.

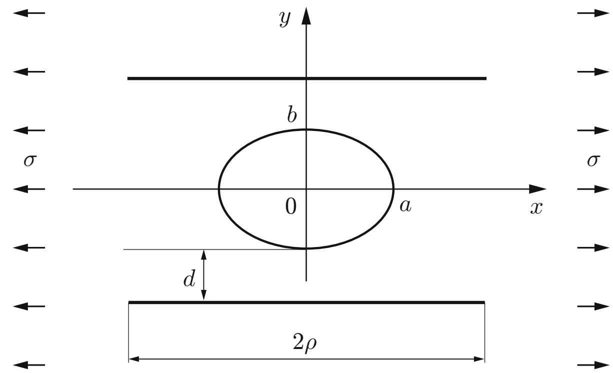

Fig. 2. Infinite anisotropic plate with an elliptic hole L_0 and two symmetric thin rigid inclusions under uniaxial tension.

To determine the unknown rotations ε_j $(j = \overline{1,k})$, one should solve system (6), (7) subject to the conditions that the principal moment of the forces acting on each rigid inclusion vanishes. These conditions are given by [3]

$$2\,\mathrm{Re}\Big(\int\limits_{L_j}(\tau_1 - \tau_2 A_0 - \bar{\tau}_2 \bar{B}_0)\Omega_1(\tau)\,d\tau_1\Big) = 0 \qquad (j = \overline{1,k}\,). \tag{8}$$

Using the singular integral equation (6) subject to conditions (7) and (8), one obtains the solution of the above-formulated problem of an anisotropic plate with an elliptic hole and thin, absolutely rigid curvilinear inclusions.

We seek a solution of the integral equation (6) in the form

$$\Omega_1(\tau) = \chi^j(\xi)(1-\xi^2)^{-1/2}, \qquad \tau \in L_j = \{\tau = \tau^j(\xi)\colon |\xi| < 1\}, \tag{9}$$

where $\chi^j(\xi)$ are bounded Hölder-continuous functions on the segment $[-1, 1]$. According to the assumptions of smooth curves L_j, the solution of Eq. (6) subject to the supplementary constraints (7) and (8) in the class of functions (9) exists and is unique [4]. With the use of the Gauss–Chebyshev quadrature formulas [5], the singular integral equations (6) with the supplementary conditions (7) and (8) is reduced to a system of linear algebraic equations for approximate values of the desired functions $\chi^j(\xi)$ at the Chebyshev nodes:

$$\xi_i = \cos\Big(\frac{2i-1}{2N_j}\,\pi\Big) \qquad (i = \overline{1,N_j}, \quad j = \overline{1,k}\,)$$

(N_j is the number of the Chebyshev nodes at the contour L_j). Theoretical estimates of the convergence of this numerical method are given in [6].

After the system of linear algebraic equations is solved and the functions $\chi^j(\xi)$ are determined, one can calculate the values of the potentials $\Phi_\nu(z_\nu)$ and stresses by formula (2) and evaluate the mode I and II stress intensity factors (SIFs) at the tip of the rigid inclusion L_j:

$$K_1(c) = \lim_{t\to c}\sigma_n\sqrt{2\pi r}, \qquad K_2(c) = \lim_{t\to c}\tau_n\sqrt{2\pi r}. \tag{10}$$

Here t is the point lying on the continuation of the rigid inclusion along the tangent at the tip c; $r = |t - c|$.

Below, calculation results are given for an anisotropic (orthotropic) plate with an elliptic hole and two symmetric thin rigid inclusions (Fig. 2). For $\varphi = 0$ (φ is the angle between the principal anisotropic direction E_1 and the x axis), the anisotropic material parameters are as follows: $E_1 = 53.84$ GPa, $E_1/E_2 = 3$, $G_{12} = 8.63$ GPa, and $\nu_1 = 0.25$. The plate is subjected to a uniform tensile load $\sigma_x^\infty = \sigma$ along the x axis. Figures 3 and 4 show the normalized stress concentration factor K^* [$K^* = \sigma_x(0,b)/\sigma_x^0(0,b)$, where $\sigma_x(0,b)$ and $\sigma_x^0(0,b)$ are the stresses σ_x at the point $(0,b)$ of the elliptic hole in the presence and absence of rigid inclusions, respectively] versus the normalized lengths of the ligament d/b and the inclusion ρ/b for various values of the ellipticity parameter of the elliptic hole $\lambda = a/b$ for an orthotropic plate ($E_1/E_2 = 3$) and $\varphi = 0$. For $\lambda = 0.001$, the elliptic hole degenerates into a rectilinear cut. In this case, the calculation results agree with those given in [2] for the problem of an anisotropic plate with cuts reinforced by thin elastic ribs [for the normalized stiffness parameter of the ribs

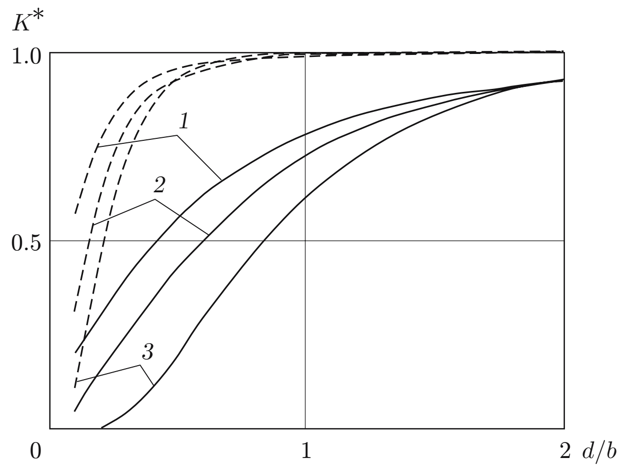

Fig. 3

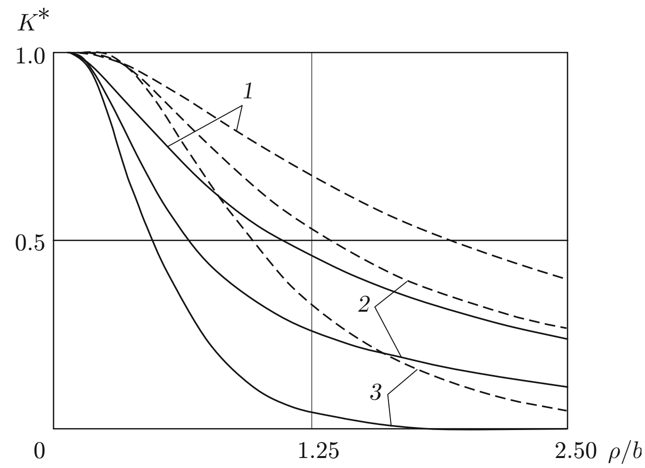

Fig. 4

Fig. 3. Factor K^* versus normalized ligament length d/b for an orthotropic plate: solid curves refer to $\rho/b = 2$ and dashed curves refer to $\rho/b = 0.5$; curves 1 refer to $\lambda = 0.001$, curves 2 to $\lambda = 1$, and curves 3 to $\lambda = 1000$.

Fig. 4. Factor K^* versus normalized length of rigid inclusions ρ/b: solid curves refer to $d/b = 0.2$ and dashed curves refer to $d/b = 0.4$; curves 1 refer to $\lambda = 0.001$, curve 2 to $\lambda = 1$, and curve 3 to $\lambda = 1000$.

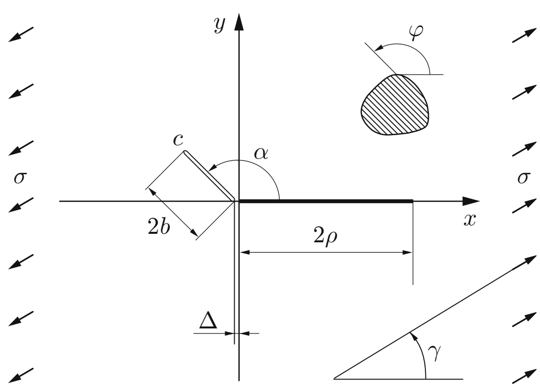

Fig. 5. Infinite anisotropic plate with an arbitrarily oriented crack emanating from the tip of a rigid linear inclusion located along the x axis under uniaxial tension.

$u^0 = E_1 bh/(E^0 F^0) = 0$, where E^0 and F^0 are the Young modulus and cross-sectional area of the rib, respectively]. One can see from Figs. 3 and 4 that the lengths of the rigid inclusion and hole-inclusion ligament have a pronounced effect on the stress concentration in the plate.

We consider the problem of determining the stress–strain state of an infinite anisotropic plate with an arbitrarily oriented crack ($\lambda = 0$): $L_0 = \{\tau(\beta) = -\Delta + b(1-\beta)\,\mathrm{e}^{i\alpha}\colon\ -1 < \beta < 1\}$ and a rigid rectilinear inclusion: $L_1 = \{\tau(\xi) = \rho(1+\xi)\colon\ -1 < \xi < 1\}$. The plate is loaded at infinity by a tensile forces σ directed at an angle γ to the x axis (Fig. 5). The calculations were performed for $\Delta/b = 0.01$, $\gamma = \pi/4$, $\alpha = \pi$, and $\varepsilon_1 = 0$ (Δ is the length of the ligament between the ends of the rigid inclusion and the crack and ε_1 is the rigid-body rotation of the inclusion). The plate material is nearly isotropic ($E_1/E_2 = 1$, $\mu_1 = 1.004i$, and $\mu_2 = 0.996i$). As in the case of a rigid inclusion, the mode I and II stress intensity factors at the crack tip c ($\beta = -1$) are calculated by formula (10). The calculated SIFs $K_{1,2}/(\sigma\sqrt{\pi\rho})$ at the crack tip c are listed in Table 1 for $\rho/b = 10$, 10^2, and 10^3 and $N = 50$ and 100 (N is the number of the Chebyshev nodes at the rigid-inclusion contour). The numerical data obtained are compared

TABLE 1

ρ/b	$K_1/(\sigma\sqrt{\pi\rho})$			$K_2/(\sigma\sqrt{\pi\rho})$		
	$N_1 = 50$	$N_1 = 100$	Data of [7]	$N_1 = 50$	$N_1 = 100$	Data of [7]
10	0.13287	0.13286	0.13285	0.20356	0.20352	0.20322
10^2	0.01592	0.01585	0.01584	0.11657	0.11657	0.11611
10^3	−0.02047	−0.02170	−0.02187	0.08807	0.09032	0.09007

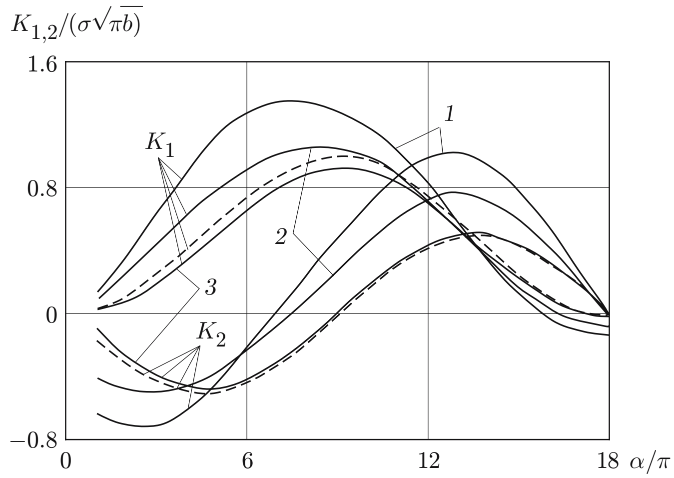

Fig. 6. Stress intensity factors $K_1/(\sigma\sqrt{\pi b})$ and $K_2/(\sigma\sqrt{\pi b})$ at the tip of an arbitrarily oriented crack near a rigid inclusion in an orthotropic plate versus the angle α for uniaxial tension: curves 1 refer to $\rho/b = 10$, curves 2 to $\rho/b = 5$, and curves 3 to $\rho/b = 1$; dashed curves refer to $\rho/b = 0.1$.

with the exact values of $K_{1,2}/(\sigma\sqrt{\pi\rho})$ at the tip of a crack emanating from the end of a rigid inclusion ($\Delta = 0$ and $\alpha = \pi$) [7]. It follows from Table 1 that for $\rho/b = 10$, the error in determining the SIFs by the numerical method is a fraction of a percent of the exact solution. For $\rho/b = 10^3$, the error is smaller than 1% even for $N = 100$. The comparison results indicate that the numerical method proposed can be effectively used to solve different-scale problems of fracture mechanics of flat structural members.

Figure 6 shows the stress intensity factors $K_1/(\sigma\sqrt{\pi b})$ and $K_2/(\sigma\sqrt{\pi b})$ at the tip c of an arbitrarily oriented crack with contour $L_0 = \{\tau(\beta) = -\Delta + b(1-\beta)(\cos\alpha + i\sin\alpha)\colon\ -1 < \beta < 1\}$ located near a rigid inclusion $L_1 = \{\tau(\xi) = \rho(1+\xi)\colon\ -1 < \xi < 1\}$ versus the angle of inclination α. Calculations were performed for $\Delta/b = 0.01$ and $\varepsilon_1 = 0$. The plate material is orthotropic ($E_1/E_2 = 3$); $\varphi = \alpha$ (see Fig. 5). At infinity, the plate is loaded by forces $\sigma_x^\infty = \sigma$ ($\gamma = 0$). One can see from Fig. 6 that the values of $K_{1,2}/(\sigma\sqrt{\pi b})$ at the crack tip increase with the rigid-inclusion length. The reason is that for small values of b/ρ, the crack is in a strongly perturbed stress field that occurs near the rigid inclusion.

It follows from the results presented that the anisotropy parameters of the plate material, the degree of ellipticity of the elliptic hole, the geometry of rigid inclusions have a considerable effect on the stress concentration in the plate. The numerical data given in Table 1 show that the integral-equation method is very effective and can be used to solve different-scale problems of determining the stress–strain state and estimate the strength of anisotropic plates with an elliptic hole (crack) and thin rigid inclusions.

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